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L'Hospital Rule:

consider the limit like: $\lim_{x\to a}\frac{f(x)}{g(x)}$, and when $x\to a$, $f(x), g(x)\to 0$, (or $f(x), g(x)\to\infty$)

(a can be ∞)

so that's our $\frac{0}{0}$ form.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists, that's important! we would have:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Ex. $$\lim_{x\to+\infty}\frac{x+\cos x}{x+\sin x} \overset{\text{L'Hospital}}{=\!=} \lim_{x\to+\infty}\frac{1-\sin x}{1+\cos x} \quad (1)$$

we have to stop in (1) for $x\to+\infty$, $\lim_{x\to+\infty}\sin x$ and $\lim_{x\to+\infty}\cos x$ doesn't exist.

so our L'Hospital Rule fails.

But consider the original limit, $\lim_{x\to+\infty}\frac{x+\cos x}{x+\sin x} = \lim_{x\to+\infty}\frac{1+\frac{\cos x}{x}}{1+\frac{\sin x}{x}} = 1$

for $|\sin x|\le 1$, $|\cos x|\le 1$.

that's why is important to check the existence of $\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Q1. $\lim_{x\to+\infty}(1+\frac{1}{x})^{x^2}e^{-x}$

First use "ln" to simplify the formula:

$$\ln(1+\tfrac{1}{x})^{x^2}e^{-x} = x^2\ln(1+\tfrac{1}{x}) - x \quad (2)$$

then use $t=\frac{1}{x}$ as substitition, so $t\to 0$

$$(2)\Rightarrow \lim_{t\to 0}\left(\frac{\ln(1+t)}{t^2}-\frac{1}{t}\right) = \lim_{t\to 0}\frac{\ln(1+t)-t}{t^2}$$

$$\text{L'Hospital} = \lim_{t\to 0}\frac{\frac{1}{1+t}-1}{2t}$$

$$= -\frac{1}{2}$$

So the result for $\lim_{x\to+\infty}(1+\frac{1}{x})^{x^2}e^{-x}$ should be $e^{-\frac{1}{2}}$

for we used the "ln".

Q2.

$$\lim_{x\to 0}\left[\frac{1}{\sin^2 x}-\frac{1}{x^2}\right]$$

I have mentioned that this problem is a bad example of using the L'Hospital Rule. if we apply rule to such form:

$$\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{\sin^2 x\cdot x^2}\right)$$, very complicated

But actually for a limit, we can have different formula, that's important in simplify the computation. For this problem:

$$\lim_{x\to 0}\frac{1}{x^2}\left(\frac{x^2}{\sin^2 x}-1\right)=\lim_{x\to 0}\frac{\left(\frac{x^2}{\sin^2 x}-1\right)}{x^2}$$ (use L'Hospital rule here)

$$=\lim_{x\to 0}\frac{2\frac{x}{\sin x}\cdot\frac{\sin x-x\cdot\cos x}{\sin^2 x}}{2x}$$

$$=\lim_{x\to 0}\frac{\sin x-x\cdot\cos x}{\sin^3 x}$$ $\left(\frac{0}{0}\right.$, once more$)$

$$=\lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{3\sin^2 x\cdot\cos x}$$

$$=\lim_{x\to 0}\frac{x}{3\sin x\cdot\cos x}=\frac{1}{3}$$

that's important to choose suitable form to apply the L'Hospital Rule.

Taylor thm:

If $f(x)\in C^n(a,b)$ which means $f(x)$ has nth continuous derivative, and $f^{(n+1)}(x)$ exists, then for $x, x_0\in(a,b)$, we can have:

$$f(x)=\underbrace{f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x)}+\underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{E_n(x)}$$

$P_n(x)$ is our taylor polynomial, $E_n(x)$ is our error term.

So we can see the idea behind the taylor thm is just use the polynomial to approximate the $f(x)$.

when we try to do the expanding, we need to consider 2 things:

(1) the order n we need to use, this is decided by $f(x)$ itself, for if $f(x)$ is just m times differentiable then $n \le m$.

(2) the point x_0 we choose to expanding.

Now we try to use Taylor thm to solve the limit we do before.

Q3. similarly we try to compute:

$$\lim_{t\to 0} \frac{\ln(1+t)-t}{t^2} \qquad (3)$$

we try to expand $\ln(1+t)$ to be some polynomials.

$\ln(1+t) = f(0) + \frac{f'(0)}{1} t + \frac{f''(0)}{2!} t^2 + O(t^3)$, the big "O" notation means has same order

$f(0) = \ln(1+0) = 0$, $f'(0) = \frac{1}{1+0} = 1$, $f''(0) = -\frac{1}{(1+0)^2} = -1$.

$\Rightarrow \ln(1+t) = t - \frac{1}{2}t^2 + O(t^3)$

$$(3) \Rightarrow \lim_{t\to 0} \frac{\ln(1+t)-t}{t^2} = \lim_{t\to 0} \frac{t-\frac{1}{2}t^2+O(t^3)-t}{t^2} = \frac{1}{2} + \lim_{t\to 0} O(t) = \frac{1}{2}.$$

that's because $mt \le O(t) \le Mt$, so $\lim_{t\to 0} O(t) = 0$.

so we can see when we use Taylor polynomial to compute the limit, we just try to estimate the order between the fractor and denominator.

Q4. $\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$.

$$\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{(x+\sin x)(x-\sin x)}{x^2\sin^2 x} \quad (4)$$

Expand the $\sin x$ at $x=0$, we have:

$$\sin x = x-\frac{1}{3!}x^3+O(x^5).$$

so $x+\sin x = 2x-\frac{1}{3!}x^3+O(x^5)= x(2-\frac{1}{6}x^2+O(x^4))$

$x-\sin x = \frac{1}{6}x^3+O(x^5) = x^3(\frac{1}{6}+O(x^2))$.

$\Rightarrow (x-\sin x)(x+\sin x) = x^4(2-\frac{1}{6}x^2+O(x^4))(\frac{1}{6}+O(x^2))$

so the coefficient of x^4 is just $2\cdot\frac{1}{6}=\frac{1}{3}$, all other are high-er order term.

for $x^2\sin^2 x = x^2(x-\frac{1}{3!}x^3+O(x^5))^2 = x^4+O(x^6)$

so (4) $\Rightarrow \lim_{x\to 0}\frac{x^4(2-\frac{1}{6}x^2+O(x^4))(\frac{1}{6}+O(x^2))}{x^4+O(x^6)}$ they have same order x^4

$=\frac{1}{3}$ equal to the quotiant of their coefficient.

Q5. Jensen inequality: an application of Taylor thm.

If $f''(x)\leq 0$ in (a,b) ($f(x)$ is convex function), try to prove:

$$\frac{1}{n}(f(x_1)+\cdots+f(x_n))\leq f\left(\frac{x_1+\cdots+x_n}{n}\right)$$

consider $f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2!}(x-x_0)^2$

$\leq f(x_0)+f'(x_0)(x-x_0)$ (for $f''(\xi)\leq 0$)

So just let $x=x_i$, $i=1,2\cdots n$. sum up then we have:

$$\sum_{i=1}^{n} f(x_i)\leq n f(x_0)+f'(x_0)\sum_{i=1}^{n}(x_i-x_0) \quad (5)$$

the only thing we left is determine the point x_0. it's clear to choose $x_0=\frac{x_1+\cdots+x_n}{n}$.

so $nx_0-(x_1+\cdots+x_n)=nx_0-\sum_{i=1}^{n}x_i=0$

(5) $\Rightarrow \sum_{i=1}^{n}f(x_i)\leq n f(x_0)$ done. similar conclusion holds when $f''(x)\geq 0$ (concave function)

MATH1010 Revision Exercise 2

From Webpage of Math 1010A.
Input here for Reference only.

1. Compute the first derivative of each of the functions below:

(a) $\arctan(x+1)$ (b) $\arcsin(x^2)$ (c) $x\arcsin(2x)$ (d) $(\arcsin(x))^2$

(e) $\dfrac{\arctan(x)}{x}$ (f) $\sqrt{\arctan(2x)}$ (g) $\arctan(\ln(x))$ (h) $\arcsin(\sqrt{1-x^2})$

2. For each of the relations below, find $\dfrac{dy}{dx}$ for the function y implicitly defined by the relation:

(a) $x = 4y - y^3$ (b) $x = y - \dfrac{1}{y}$ (c) $x = (3y+2)^{10}$ (d) $x = (4-y)(3+y^2)$

(e) $x = y^{-2}\sin(y)$ (f) $x = \sqrt{\dfrac{y+1}{y+2}}$ (g) $x^2 + y^2 = 4.$ (h) $x^2 + y^2 - 3x + 1 = 0$

(i) $4y^2 + xy - 6x^2 = 0$ (j) $2x^3 + y^3 - 3x^2y = 1$ (k) $x^2\sin(y) - y\cos(x) = 2$

(l) $x\cos(y) + y^2\sin(x) = 0$

3. Let n be a positive integer. Let $f(x) = (1-x^2)^n$ for any $x \in \mathbb{R}$.

(a) Show that $(1-x^2)f'(x) + 2nxf(x) = 0$ for any $x \in \mathbb{R}$.

(b) Show that $(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) + n(n+1)f^{(n)}(x) = 0$ for any $x \in \mathbb{R}$.

4. Let $f(x) = e^x\ln(1+x)$ for any $x \in (-1,+\infty)$.

(a) Show that $(1+x)f''(x) - (1+2x)f'(x) + xf(x) = 0$ for any $x \in (-1,+\infty)$.

(b) Let n be a non-negative integer. Show that $(1+x)f^{(n+3)}(x) + (n-2x)f^{(n+2)}(x) + (x-2n-2)f(x) + (n+1)f^{(n)}(x) = 0$ for any $x \in (-1,+\infty)$.

5. Let $f(x) = \dfrac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$ for any $x \in \mathbb{R}$.

(a) Show that $(1+x^2)f'(x) + xf(x) = 1$ for any $x \in \mathbb{R}$.

(b) Let n be a non-negative integer. Show that $(1+x^2)f^{(n+2)}(x) + (2n+3)xf^{(n+1)}(x) + (n+1)^2f^{(n)}(x) = 0$ for any $x \in \mathbb{R}$.

6. Let $f(x) = (\arcsin(x))^2$ for any $x \in (-1,1)$.

(a) Show that $(1-x^2)f'(x) - xf(x) = 2$ for any $x \in (-1,1)$.

(b) Let n be a positive integer. Show that $(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) - (n)^2f^{(n)}(x) = 0$ for any $x \in (-1,1)$.

7. Let $f : [3,6] \longrightarrow \mathbb{R}$ be a continuous function. Suppose f is differentiable on $(3,6)$, and $|f'(x) - 9| \leq 3$ on $(3,6)$. Show that $18 \leq f(6) - f(3) \leq 36$.

8. Let $\beta \in (1,+\infty)$. Let $f : (0,+\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^\beta + \beta - 1 - \beta x$ for any $x \in (0,+\infty)$.

(a) i. Compute f'.

ii. Show that f is strictly decreasing on $(0,1]$.

iii. Show that f is strictly increasing on $[1,+\infty)$.

iv. Determine whether f attains the maximum and/or the minimum on $(0,+\infty)$.

(b) Hence, or otherwise, show that $(1+r)^\beta \geq 1 + \beta r$ for any $r \in (-1,+\infty)$.

9. Prove the following inequalities:

From Webpage of Math 1010A.
Input here for Reference only.

(a) $\dfrac{x}{1+x^2} < \arctan(x) < x$ for any $x \in (0,+\infty)$.

(b) $0 < \ln(1+x) - \dfrac{2x}{2+x} < \dfrac{x^3}{12}$ for any $x \in (0, +\infty)$.

10. (a) Prove that $1 - \dfrac{x^2}{2} < \cos(x)$ for any $x \in (0, 2\pi]$.

(b) Prove that $\cos(x) < 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ for any $x \in (0, 2\pi]$.

(c) Prove that $1 - \dfrac{x^2}{2} < \cos(x) < 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ for any $x \in (2\pi, +\infty)$.

(d) Prove that $1 - \dfrac{x^2}{2} < \cos(x) < 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ for any $x \in \mathbb{R}\backslash\{0\}$.

11. Apply L'Hôpital's Rule to evaluate each of the limits below.

(a) $\displaystyle\lim_{x\to 0} \frac{x + \tan(x)}{\sin(2x)}$ (b) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin(x)}$ (c) $\displaystyle\lim_{x\to 0} \frac{\arctan(x)}{x}$ (d) $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{2\ln(1+x)}$

(e) $\displaystyle\lim_{x\to 0} \frac{x^2 + 3x + 4}{3x^3 + 5}$ (f) $\displaystyle\lim_{x\to 0} \frac{x - \sin(x)}{2x^3}$ (g) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}$ (h) $\displaystyle\lim_{x\to 1} \frac{e^{x-1} - x}{(x-1)^2}$

(i) $\displaystyle\lim_{x\to 0} \frac{24\cos(x) - 24 - 12x^2 + x^4}{\sin^6(x)}$ (j) $\displaystyle\lim_{x\to 0} \frac{x\tan(x)}{1 - \sqrt{1 - x^2}}$ (k) $\displaystyle\lim_{x\to +\infty} \frac{\ln(e^x + x^2)}{x^2}$

(l) $\displaystyle\lim_{x\to 1} \frac{1 + \ln(x) - x^x}{1 + \ln(x) - x}$ (m) $\displaystyle\lim_{x\to 0^+} \frac{(\ln(x))^5}{\sqrt[5]{x}}$ (n) $\displaystyle\lim_{x\to +\infty} \frac{\ln(1 + xe^{2x})}{\sin^2(x)}$

(m) $\displaystyle\lim_{x\to 0^+} \frac{\ln(\sin(\alpha x))}{\ln(\sin(\beta x))}$. (Here α, β are positive real numbers.)

12. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

(a) $\displaystyle\lim_{x\to 0^+} x^2 e^{(-x^{-2})}$ (b) $\displaystyle\lim_{x\to 0^+} \sin(x)\ln(x)$ (c) $\displaystyle\lim_{x\to 0^+} x\csc(2x)$ (d) $\displaystyle\lim_{x\to +\infty} x\left[\left(1 + \frac{1}{x}\right)^x - e\right]$

13. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

(e) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\arctan(x)}\right)$ (f) $\displaystyle\lim_{x\to 0^+} \left(\csc^2(x) - \frac{1}{x^2}\right)$

(g) $\displaystyle\lim_{x\to 1^+} \left(\frac{x^2}{(1-x)^2} - \frac{1}{(\ln(x))^2}\right)$ (h) $\displaystyle\lim_{x\to 0^+} \left(\cot(x) - \frac{1}{x}\right)$

14. Evaluate each of the limits below. When necessary, apply L'Hôpital's Rule.

(a) $\displaystyle\lim_{x\to +\infty} x^{\frac{1}{x}}$ (b) $\displaystyle\lim_{x\to 0^+} x^{\sin(x)}$ (c) $\displaystyle\lim_{x\to 0^+} \left(\ln\left(\frac{1}{x}\right)\right)^x$ (d) $\displaystyle\lim_{x\to +\infty} \left(1 - \frac{1}{x}\right)^{-x}$

(e) $\displaystyle\lim_{x\to +\infty} \left(1 + \frac{2}{x}\right)^{-x}$ (f) $\displaystyle\lim_{x\to +\infty} \left(1 + \frac{3}{x^2}\right)^x$ (g) $\displaystyle\lim_{x\to +\infty} \left(\frac{x+1}{x-1}\right)^x$ (h) $\displaystyle\lim_{x\to +\infty} \left(\frac{x^2+1}{x^2-1}\right)^{x^2}$

(i) $\displaystyle\lim_{x\to +\infty} \left(\frac{x^2 - 2x - 3}{x^2 - 3x - 28}\right)^x$ (j) $\displaystyle\lim_{x\to \frac{\pi}{2}^-} (\tan(x))^{\cos(x)}$ (k) $\displaystyle\lim_{x\to 0^+} (1 + \sin(x))^{\frac{1}{x}}$

(l) $\displaystyle\lim_{x\to 0^+} (1 + \sin^2(x))^{\frac{1}{x}}$ (m) $\displaystyle\lim_{x\to 1^+} x^{\frac{e^x}{1-x}}$ (n) $\displaystyle\lim_{x\to 0^+} (1 - \cos(x))^{\frac{1}{\ln(x)}}$

(o) $\displaystyle\lim_{x\to \frac{\pi}{2}^-} (\cos(x))^{\ln(\sin(x))}$ (p) $\displaystyle\lim_{x\to 0} \left(\frac{\arcsin(x)}{x}\right)^{\frac{1}{x^2}}$ (q) $\displaystyle\lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^{\frac{1}{x^2}}$

15. Evaluate the each of the limits below. Think carefully whether to apply L'Hôpital's Rule or not.

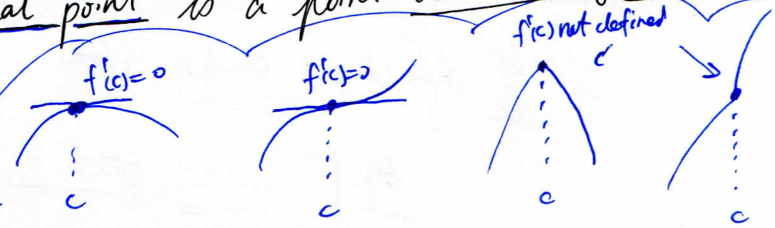
(a) $\displaystyle\lim_{x\to +\infty} \frac{x + \sin(x)}{x - \sin(x)}$ (b) $\displaystyle\lim_{x\to +\infty} \frac{e^x + x\sin(x) + \cos(x)}{e^x + \cos(x)}$ (c) $\displaystyle\lim_{x\to +\infty} \frac{x^2 + \sin(2x)}{(2x^3 + x + \sin(x))e^{\sin(x)}}$

Notice: A test next week (on lectures, not on tutorial), everything up to Cauchy's mean value theorem, & L'Hôpital's rule

PLAN: Go through revision exercise 2 (on webpage of Math 1010 a);

Note: Critical points

In Thomas' calculus, a critical point is a point whether $f'(c)=0$, or $f(c)$ is not defined.



are all critical points!

Note: Critical points

$\Rightarrow$ suspects of relative extrema;

$\Updownarrow$ local min. / max.

- 1st derivative test,
- 2nd derivative test, etc

whether they are relative extreme

Math 1010 Revision Exercise 2

1. Compute 1st derivative ... ; (omitted here);

2. $\frac{dy}{dx}$ for y implicitly defined by relations;

Sol'n: $\boxed{\text{Slogan}}$ Differentiate the equation, pretend y is a function of x, Using chain rule. $\Rightarrow$ Get $\frac{dy}{dx}$.

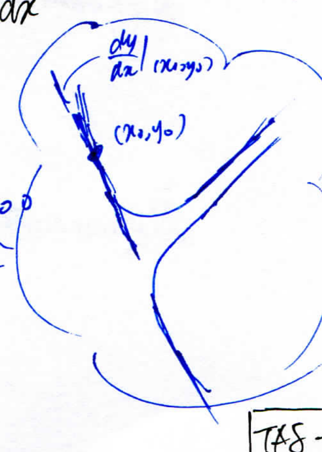
e.g. (j) $2x^3+y^3-3x^2y=1$.

Differentiate both sides $\frac{d}{dx}$, get $6x^2+3y^2\cdot\frac{dy}{dx}-6xy-3x^2\frac{dy}{dx}=$

$$\Rightarrow \frac{dy}{dx}=\frac{6x^2-6xy}{3x^2-3y^2}=\frac{2x^2-2xy}{x^2-y^2};$$

means, $\forall (x_0,y_0)\in\{2x^3+y^3-3x^2y=1\}\subset\mathbb{R}^2$,

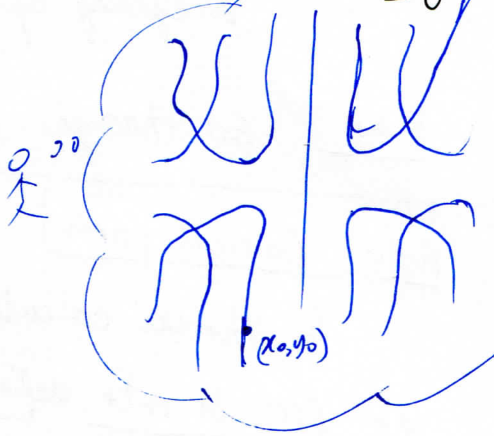
$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)}=\frac{2x_0^2-2x_0y_0}{x_0^2-y_0^2}; \qquad \square$$



(l). $x\cos(y) + y^2\sin(x) = 0$;

Differentiate both sides $\frac{d}{dx}$: $\cos(y) - x\sin(y)\cdot\frac{dy}{dx} + 2y\frac{dy}{dx}\sin x + y^2\cos x = 0$

Hence $$\frac{dy}{dx} = \frac{\cos y - y^2\cos x}{x\sin y - 2y\sin x};$$



means that, $\forall (x_0,y_0) \in \{x\cos(y) + y^2\sin(x) = 0\}$.

if $\frac{dy}{dx}$ defined at (x_0,y_0), then

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \frac{\cos y_0 - y_0^2\cos x_0}{x_0\sin y_0 - 2y_0\sin x_0};$$

$\square$

3. $n \in \mathbb{Z}_+$, $f(x) = (1-x^2)^n$ for any $x \in \mathbb{R}$.

(a) Show that $(1-x^2)f'(x) + 2nxf(x) = 0$, $\forall x \in \mathbb{R}$; $(*)$

(b) Show that $(1-x^2)f^{(n+2)}(x) - \cancel{(2n+1)}\, xf^{(n+1)}(x) + n(n+1)f^{(n)}(x) = 0$, $\forall x \in \mathbb{R}$;

$\downarrow$ $-2xf^{(n+1)}(x)$

Sol'n: (a) Chain rule: $f'(x) = n(1-x^2)^{n-1}\cdot(-2x)$;

Hence $(1-x^2)f'(x) = -2x\cdot n\cdot\underbrace{(1-x^2)^n}_{f(x)} \Rightarrow$ done;

(b) Method:
- Whether compute $f^{(n)}(x)$, or
- Use induction: relations between $f^{(k+2)}, f^{(k+1)}, f^{(k)}$? $k = 0, 1, \cdots, n$.

Differentiate $(*)$: $(1-x^2)f' + 2nx\cdot f = 0$

$\frac{d}{dx} \rightsquigarrow (-2x)f' + (1-x^2)f'' + 2nf + 2nx\cdot f' = 0$

$\Rightarrow (1-x^2)f'' + (2n-2)xf' + 2nf = 0$; $(k=0)$

Differentiate again:

$\frac{d}{dx} \rightsquigarrow (-2x)f'' + (1-x^2)f''' + (2n-2)f' + (2n-2)xf'' + 2nf' = 0$

$\Rightarrow (1-x^2)f''' + (2n-4)xf'' + (4n-2)f' = 0$; $(k=1)$

Inductively, we get $\quad (1-x^2) f^{(k+2)} + (2n - 2(k+1)) x f^{(k+1)} + \sum_{j=0}^{k} (2n-2j) \cdot f^{(k)} = 0 \quad (**)$

$\hookrightarrow \begin{cases} k=0,1 \ \checkmark ; \\ k \checkmark \Rightarrow (k+1) \checkmark, \text{ just differentiate above } (**). \end{cases}$

Hence when $k=n$, we get

$$(1-x^2) f^{(n+2)}(x) + (-2x) f^{(n+1)}(x) + n(n+1) f^{(n)}(x) = 0$$

$\square$

Remark: We can directly use the generalized Leibniz rule:

$$(u \cdot v)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)} ;$$

(exercise: prove above formula by induction on n)

Hence from $\quad (1-x^2) f'(x) = -2nx f(x)$;

use Leibniz rule, take $(n+1)$-th order derivative of both sides:

$$(1-x^2) f^{(n+2)}(x) - (n+1) 2x \cdot f^{(n+1)}(x) \underbrace{-2\binom{n+1}{2}}_{= -n(n+1)} f^{(n)}(x) = -2nx f^{(n+1)}(x) \underbrace{-2n}_{-2n(n+1)} \overset{\cdot (n+1)}{} f^{(n)}(x)$$

$$\Rightarrow \quad (1-x^2) f^{(n+2)}(x) - 2x f^{(n+1)}(x) + n(n+1) f^{(n)}(x) = 0.$$

$\square$

Summary: How to deal with higher derivatives? Basic tools:

(A) Basic formula:

- $(x^k)^{(n)} = \begin{cases} k(k-1)\cdots(k-n+1) \cdot x^{k-n} & ; \ n \leq k ; \\ 0 & ; \ n > k ; \end{cases}$

- $(e^x)^{(n)} = e^x ; \quad (a^x)^{(n)} = (\ln a)^n \cdot a^x ;$
- $(\ln x)^{(n)} = (-1)^{n-1} (n-1)! \, x^{-n} ;$
- $(\sin x)^{(n)} = \sin(x + \frac{n\pi}{2}) ;$

  $(\cos x)^{(n)} = \cos(x + \frac{n\pi}{2}) ;$

- & useful one: $\quad a, b \in \mathbb{R}$;

  $(f(ax+b))^{(n)} = a^n f^{(n)}(ax+b) ;$

(B) Observation + Induction!

(C). Leibniz rule: $(u\cdot v)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} \cdot v^{(n-k)}$;

[(D). Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$;

But if you can express $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, then

$f^{(n)}(a) = n!\, a_n$;]

Now for 4~6, the method is always:

(i) Whether compute directly f', f'', or use relation of f & take derivatives $\Rightarrow$ Get recursion formula

$$(*) \begin{cases} a(x) f' + b(x) f = c(x), \text{ or} \\ a f'' + b f' + c\cdot f' = d; \end{cases}$$

(ii) Differentiate $(*)$ n-times (or (n+1)-times, (n+2)-times, ...)

to get relations of $f^{(n+2)}$, $f^{(n+1)}$, $f^{(n)}$, ...

How? $\begin{cases} \bullet \text{ (B). Observation + Induction;} \\ \bullet \text{ (C). Leibniz rule;} \end{cases}$

4. $f(x) = e^x \ln(1+x)$;

(a) $f'(x) = \underbrace{e^x \ln(1+x)}_{f(x)} + e^x \cdot \frac{1}{1+x} \Rightarrow (1+x) f' = (1+x) f + e^x$;

Differentiate again: $(1+x) f'' + f' = (1+x) f' + f + \overbrace{e^x}^{(1+x)f' - (1+x)f}$

$$\underline{(1+x) f'' - (1+2x) f' + x f = 0}, \quad (*) \checkmark$$

(b) Differentiate $(*)$ (n+1)-times; induction, or Leibniz:

$$(1+x) f^{(n+3)} + \underbrace{(n+1) f^{(n+2)} - (1+2x) f^{(n+2)}}_{(n-2x) f^{(n+2)}} \underbrace{- (n+1)\cdot 2\cdot f^{(n+1)} + x\cdot f^{(n+1)}}_{(x-2n-2) f^{(n+1)}} + (n+1) f^{(n)} =$$

7. $f(x) = \dfrac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$ for any $x \in \mathbb{R}$.

(a) $\sqrt{1+x^2}\, f(x) = \ln(x+\sqrt{1+x^2})$.

Differentiate, $\dfrac{2x}{2\sqrt{1+x^2}} f(x) + \sqrt{1+x^2}\, f'(x) = \underbrace{\dfrac{1+\dfrac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}}_{=\frac{1}{\sqrt{1+x^2}}}$

$\Rightarrow \quad (1+x^2) f'(x) + x f(x) = 1. \quad (*)$

(b) Differentiate $(*)$ $(n+1)$ times, using observation & induction. OR

use Leibniz rule $\underbrace{\left((1+x^2) f'(x)\right)^{(n+1)}}_{\parallel} + \underbrace{\left(x f(x)\right)^{(n+1)}}_{\parallel} = 0$

$\underbrace{(1+x^2) f^{(n+2)} + \binom{n+1}{1}(1+x^2)' f^{(n+1)} + \binom{n+1}{2}(1+x^2)'' f^{(n)}}_{\parallel}$ $\quad \underbrace{x f^{(n+1)} + \binom{n+1}{1} x' f^{(n)}}_{\parallel}$

$(1+x^2) f^{(n+2)} + 2(n+1)x f^{(n+1)} + 2\cdot\dfrac{(n+1)\cdot n}{2} f^{(n)}$ $\quad x\cdot f^{(n+1)} + (n+1) f^{(n)}$

$\Rightarrow (1+x^2) f^{(n+2)} + (2n+3)x f^{(n+1)} + (n+1)^2 f^{(n)} = 0$ $\square$

$f(x) = (\arcsin(x))^2$ for any $x \in (-1,1)$.

(a) $f'(x) = 2\arcsin(x)\cdot\dfrac{1}{\sqrt{1-x^2}}$, then

$(1-x^2) f'^2(x) = 4 f(x)$; differentiate again, get

$-2x\cdot f'^2(x) + (1-x^2)\, 2 f' \cdot f'' = 4 f'$. since $f' \neq 0$. (as long as $x \neq 0$).

$\Rightarrow \quad (1-x^2) f'' - x f' = 2$;

(b) Use Leibniz rule, differentiate both sides n-times:

$(1-x^2) f^{(n+2)}(x) - 2nx f^{(n+1)}(x) - n(n-1) f^{(n)}(x) - x f^{(n+1)}(x) - n\cdot f^{(n)}(x) =$

$\Rightarrow (1-x^2) f^{(n+2)}(x) - (2n+1)x f^{(n+1)}(x) - n^2 f^{n}(x) = 0$ $\square$

TA5-2

$\cancel{7}$, ~~9, 10~~ : Lagragian Mean Value thm.

9,10 : See proof later.

Recall: Thm (Lag. Mean Value thm). $f(x)$ is cont. $[a,b]$, differentiable on (a,b). Then $\forall\, x_1 \neq x_2 \in [a,b]$, $\exists\, \xi$ $\underbrace{\text{between } x_1 \text{ and } x_2}$, s.t.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi). \quad (\star) \qquad \begin{cases} x_1 < \xi < x_2 & \text{or} \\ x_2 < \xi < x_1 \end{cases}$$

$\#$.

If we write $(\star)$ in following form, assume $x_1 < x_2$,

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1), \qquad \xi \in (x_1, x_2);$$

the difference of $f(x_2)$ & $f(x_1)$ is controled by $|f'(\xi)| \cdot (x_2 - x_1)$;

Very useful. (e.g. if $f' \equiv 0$ on (a,b), $\Rightarrow f$ const. on $[a,b]$, since for fixed x_0, $\forall\, x \in [a,b] \setminus \{x_0\}$

$$f(x) - f(x_0) = \underbrace{f'(\xi)}_{=0} (x - x_0), \quad \xi \in (x_0, x) \text{ or } (x, x_0)$$

$\square$)

7. f cont. $[3,6]$, diff on $(3,6)$. $|f'(x) - 9| \le 3$ on $(3,6)$;

Show $18 \le f(6) - f(3) \le 36$.

Prove: Lagragian mean value thm,

$$f(6) - f(3) = f'(\xi) \cdot \underbrace{(6-3)}_{3}, \qquad \xi \in (3,6)$$

But $|f'(\xi) - 9| \le 3 \iff 6 \le f'(\xi) \le 12$;

Hence

$$18 \le f(6) - f(3) \le 36.$$

$\square$

(~~omit this method~~)

Ignore this method for the time being; See the proof on next next page.

10. $1 - \frac{x^2}{2} < \cos(x) < 1 - \frac{x^2}{2} + \frac{x^4}{24}, \quad \forall\, x \in (0, 2$

ignore this for now

Use Taylor's thm: f is cont. on (a,b), & have $(n+1)$ derivatives, then

$$f(x) = \underbrace{f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n}_{TP_n(x)} + \underbrace{\frac{f^{(n+1)}(\xi)(x-c)^{n+1}}{(n+1)!}}_{E_n(x)}$$

Since $(\cos(x))^{(n)} = \cos(x + \frac{n\pi}{2})$. Hence

Ex10 : Ignore this method for the time being.
See another proof on next page.

$$(\cos x)'(0)\cdot x + \dots + \frac{\cos(x+\frac{n}{2}\pi)\big|_{x=0}}{n!}\cdot x^n + \frac{\cos(\xi+\frac{n+1}{2}\pi)\,x^{n+1}}{(n+1)!}$$

Hence $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots + (-1)^m \frac{x^{2m}}{(2m)!} + r_{2m+1}(x)$ $\quad \xi \in (0,x)$ or $(x,0)$:

where $r_{2m+1}(x) = (-1)^{m+1} \cos(\theta x) \cdot \frac{x^{2m+2}}{(2m+2)!}$, $\quad 0<\theta<1$.

- $\cos x = 1 - \frac{x^2}{2} + \cos(\xi)\cdot\frac{x^4}{4!}$, $\quad \xi \in (0, x)$;

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cos(\xi)\cdot\frac{x^6}{6!}$; $\quad \xi \in (0,x)$;

Hence for $x \in (0,\pi)$, $\cos(\xi) > 0$, we get

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}, \quad x \in (0,\pi);$$

Now consider $x \in [\pi, 2\pi]$, since

$$\left(\cos x - (1-\frac{x^2}{2})\right)' = \underbrace{-\sin x + x}_{} > 0 \text{ for } x \in [\pi, 2\pi];$$

$$\left(\cos x - (1-\frac{x^2}{2}+\frac{x^4}{24})\right)' = -\sin x + x - \frac{x^3}{3!} < 0, \text{ for } x \in [\pi, 2\pi];$$

Ex10 : Ignore this method for the time being.
See the proof below and next page.

Another method: <u>using derivatives to prove inequality.</u> (9, 10).

<u>FACT</u>: • (a) if $f(x_0) \geq g(x_0)$, & $f'(x) - g'(x) \geq 0$, $\forall x \in (x_0, b]$.

$\Rightarrow f(x) \geq g(x)$, $\forall x \in [x_0, b]$;

• (b) if $f(x_0) \geq g(x_0)$ & $f'(x) - g'(x) > 0$, $\forall x \in (x_0, b)$;

$\Rightarrow f(x) > g(x)$, $\forall x \in (x_0, b]$.

<u>Reason</u>: For $h(x) = f(x) - g(x)$, use Lagrangian Mean Value thm:

$h(x) - h(x_0) = h'(\xi)(x - x_0)$, since $h'(\xi) = f'(\xi) - g'(\xi)$ ≥ 0, in (a); > 0, in (b); $\xi \in (x_0, x)$.

$\Rightarrow h(x) = \overset{\geq 0}{h(x_0)} + \overset{\geq 0 \,\&\, > 0}{h'(\xi)(x - x_0)}$ ≥ 0 in (a); > 0 in (b)

$\square$

9. (a) $\underbrace{\frac{x}{1+x^2}}_{f(x)} < \underbrace{\arctan(x)}_{g(x)} < \underbrace{x}_{h(x)}$, $\forall x \in (0,+\infty)$;

Proof: since when $x=0$, $f(0)=g(0)=h(0)=0$;

& $f'(x) = \frac{(1+x^2) - x\cdot(2x)}{1+x^2} = \frac{1-x^2}{1+x^2}$;

$g'(x) = \frac{1}{1+x^2}$;

$h'(x) = 1$;

& $\underbrace{\frac{1-x^2}{1+x^2} < \frac{1}{1+x^2}}_{(\Leftrightarrow\ 1-x^2<1)} < 1$, $\forall x \in (0,+\infty)$ $\square$

$(1+x^2>1;)$

(b) $0 < \underbrace{\ln(1+x) - \frac{2x}{2+x}}_{f(x)} < \underbrace{\frac{x^3}{12}}_{g(x)}$. $x \in (0,+\infty)$

Proof: $f(0)=g(0)=0$, $f'(x) = \frac{1}{1+x} - \frac{2(2+x)-2x}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$

$= \frac{x^2}{(1+x)(2+x)^2}$;

$g'(x) = \frac{x^2}{4}$; now $\forall x>0$, show $0 < \frac{x^2}{(1+x)(2+x)^2} < \frac{x^2}{4}$; $\square$

10. (a) $\underbrace{1-\frac{x^2}{2}}_{f(x)} < \underbrace{\cos x}_{g(x)}$ $\forall x \in (0,2\pi]$.

Proof: Method: Use ~~(b)~~ Fact again and again!

$f(0)=g(0)=1$, $f'(x) = 0 - x$; $g'(x) = -\sin x$;

$f'(0)=g'(0)=0$, $f''(x) = -1$; $g''(x) = -\cos x$, $x \in (0,2\pi)$;

since $x \in (0,2\pi)$, $\underset{f''(x)}{-1} < \underset{g''(x)}{-\cos x}$, $\Rightarrow$ $\underset{f'}{-x} < \underset{g'}{-\sin x}$, $\forall x \in (0,2\pi)$

$\Rightarrow 1-\frac{x^2}{2} < \cos x$, $\forall x \in (0,2\pi]$ $\square$

(b) $\underbrace{\cos(x)}_{f(x)} < \underbrace{1-\frac{x^2}{2}+\frac{x^4}{24}}_{g(x)}, \quad \forall x \in (0, 2\pi]$.

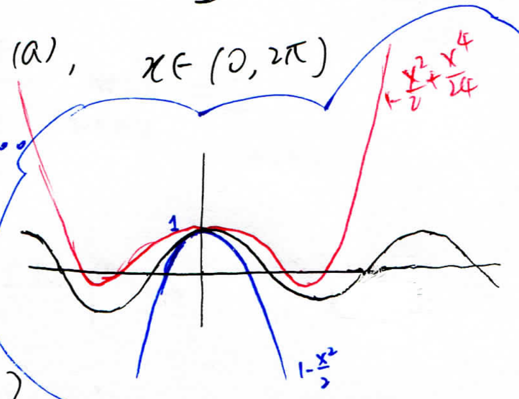
Proof:

- $f(0) = g(0) = 1$; $f'(x) = -\sin x$, $g'(x) = -x + \frac{x^3}{6}$;
- $f'(0) = g'(0) = 0$; $f''(x) = -\cos x$, $g''(x) = -1 + \frac{x^2}{2}$;

Now $f''(x) < g''(x)$ as proved in (a), $x \in (0, 2\pi)$

$\Rightarrow f'(x) < g'(x)$, $x \in (0, 2\pi]$

$\Rightarrow f(x) < g(x)$, $x \in (0, 2\pi]$.



(c), • $1-\frac{x^2}{2} \searrow x \in [\sqrt{2}, +\infty)$, $\Rightarrow 1-\frac{x^2}{2} \leq 1-2\pi^2 < -17 < -1 \leq \cos x$, $x \in [\sqrt{2}, +\infty)$;

(d). They are all even f'ns!

• $1-\frac{x^2}{2}+\frac{x^4}{24} \nearrow$, $x > \sqrt{6}$

or $x > \sqrt{12} \Rightarrow 1-\frac{x^2}{2}+\frac{x^4}{24} > 1 \geq \cos x$;

L'Hôpital's Rule

(Recall from last tutorial). If f & g differentiable on $(a,b) \setminus \{c\}$; &

- $f(c) = g(c) = 0$; (or $\lim_{x\to c} f(x) = \infty$, $\lim_{x \to c} g(x) = \infty$)
- $g'(x) \neq 0$, for x near c; $x \neq c$;
- $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ $\exists$ & $= L$, (finite number);

Then $\lim_{x \to c} \frac{f(x)}{g(x)}$ $\exists$ & $= L$.

Rmk: check all the conditions: $\frac{0}{0}$ or $\frac{\infty}{\infty}$; & $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ $\exists$;

WARNING: if $\lim_{x\to c} \frac{f'(c)}{g'(c)}$ does not exist, can not imply $\lim_{x\to c} \frac{f(c)}{g(c)}$ does not exist; can only imply L'Hôpital's Rule does not work here.

(e.g. 15 (a) $\lim_{x\to+\infty} \frac{x+\sin(x)}{x-\sin x}$)

15. (a) $\lim_{x \to +\infty} \frac{x+\sin(x)}{x-\sin(x)}$;

Sol'n: 1^{st}, $\lim_{x\to+\infty} \frac{1+\frac{\sin(x)}{x}}{1-\frac{\sin(x)}{x}} = 1$, since $\lim_{x\to+\infty} \frac{\sin x}{x} = 0$ ($\sin x$ bounded);

2^{nd}: Check L'Hôpital's condition: { • $\frac{\infty}{\infty}$ ✓; • $(x-\sin(x))' = 1-\cos x$ not same; • $\lim_{x\to+\infty} \frac{1+\cos x}{1-\cos x}$ does not exist!

Hence can not use L'Hôpital's rule!

(b) $$\lim_{x\to+\infty} \frac{e^x + x\sin(x) + \cos(x)}{e^x + \cos(x)} \overset{?}{=} L$$

can use L'Hôpital Here check it!

Sol'n: 1st. since $\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{x'}{(e^x)'} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$;

$$L = \lim_{x\to+\infty} \frac{1 + \frac{x}{e^x}\cdot\sin(x) + \frac{1}{e^x}\cos(x)}{1 + \frac{\cos(x)}{e^x}} = 1;$$

2nd. L'Hôpital's rule? • $\frac{\infty}{\infty}$ ✓;

• $(e^x + \cos(x))' = e^x - \sin x > 0$, for $x >> 0$; ✓

• $$\lim_{x\to+\infty} \frac{(e^x + x\sin x + \cos x)'}{(e^x + \cos x)'} = \lim_{x\to+\infty} \frac{e^x + x\cdot\cos x}{e^x - \sin x}$$

$$= \lim_{x\to+\infty} \frac{1 + \frac{x}{e^x}\cos x}{1 - \frac{\sin x}{e^x}} = 1;$$ ✓

Hence can use L'Hôpital's rule, & $L = \lim \frac{(\ )'}{(\ )'} = 1$;

although not very useful here

(c). $$\lim_{x\to+\infty} \frac{x^2 + \sin(2x)}{(2x^3 + x + \sin(x))\, e^{\sin(x)}};$$

Sandwich: $\frac{1}{2ex} \le \frac{1}{2x\, e^{\sin(x)}} \le \frac{e}{2x}$, $\frac{1}{2ex} \to 0$, $\frac{e}{2x} \to 0$

Sol'n: 1st. $$\lim_{x\to+\infty} '' = \lim_{x\to+\infty} \frac{1 + \frac{\sin(2x)}{x^2}}{2x\left(1 + \frac{1}{2x^2} + \frac{\sin(x)}{2x^3}\right) e^{\sin(x)}} = 0;$$

bounded: $0 < e^{-1} \le e^{\sin(x)} \le e$

2nd. $\frac{\infty}{\infty}$ ✓;

...... You can use L'Hôpital here (you can check it).

BUT DO YOU WANT TO ?

A little trick

In the process of using derivatives, it is always convenient to use the following substitution: (in multiplicative / quotient terms, not in additive terms!)

$x \to 0$, $\sin x \sim x \sim \tan x \sim \ln(1+x)$; & $1-\cos x \sim \frac{x^2}{2}$;

The reason is: $\lim_{x\to 0} f(x)\cdot \sin x = \lim_{x\to 0} f(x)\cdot x \cdot \frac{\sin x}{x}$ $\quad \to 1 \ (x\to 0)$

$= \left(\lim_{x\to 0} f(x)\cdot x\right)\cdot\left(\lim \frac{\sin x}{x}\right)$

$= \lim_{x\to 0} f(x)\cdot x$, provided this exists;

e.g.: 11 (i). $\lim_{x\to 0} \frac{24\cos(x) - 24 - 12x^2 + x^4}{\sin^6(x)}$

(substitution!)

$= \lim_{x\to 0} \frac{24\cos(x) - 24 - 12x^2 + x^4}{x^6} \cdot \frac{x^6}{\sin^6(x)}$ $\to 1$ $\left(\frac{0}{0}\right)$

$\overset{?}{=} \lim_{x\to 0} \frac{24\sin x - 24x + 4x^3}{6x^5}$ $\left(\frac{0}{0}\right)$ $\overset{?}{=} \lim_{x\to 0} \frac{24\cos x - 24 + 12x^2}{30x^4}$ $\left(\frac{0}{0}\right)$

$\overset{?}{=} \lim_{x\to 0} \frac{-\cancel{24}\sin x + \cancel{24}x}{\cancel{120}\,5\,x^3}$ $\left(\frac{0}{0}\right)$ $\overset{?}{=} \lim_{x\to 0} \frac{-\cos x + 1}{15x^2}$ $\left(\frac{0}{0}\right)$

$\overset{?}{=} \lim_{x\to 0} \frac{-\sin x}{30x} = -\frac{1}{30}$;

since the last limit exist, the L'Hôpital's rule's conditions are all satisfied, then above $\overset{?}{=}$ are all real $=$.

13. (f). $\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \lim_{x\to 0} \frac{x^2 - \sin^2 x}{\sin^2 x \cdot x^2}$ $\square$

$= \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^4} \cdot \frac{x^2}{\sin^2 x}$ $\to 1$ $\overset{?}{=} \lim_{x\to 0} \frac{2x - \overbrace{2\sin x \cdot \cos x}^{\sin 2x}}{4x^3}$ $\frac{0}{0}$ $\overset{?}{=} \lim_{x\to 0} \frac{2 - 2\cos 2x}{12x^2}$

$\overset{?}{=} \lim_{x\to 0} \frac{+4\sin 2x}{24x}$ $\frac{0}{0}$ $= \frac{1}{3}$; again all above $\overset{?}{=}$ are $=$; $\square$

14. (h) $\lim_{x \to +\infty} \left(\frac{x^2+1}{x^2-1} \right)^{x^2}$ (1^∞)

consider $\lim_{x\to+\infty} x^2 \ln\left(\frac{x^2+1}{x^2-1}\right) = \lim_{x\to+\infty} \frac{\overbrace{\ln\left(\frac{x^2+1}{x^2-1}\right)}^{\ln(x^2+1)-\ln(x^2-1)}}{\frac{1}{x^2}}$ $\left(\frac{0}{0}\right)$

$\overset{H}{=} \lim_{x\to+\infty} \frac{\frac{2x}{x^2+1} - \frac{2x}{x^2-1}}{-\frac{2}{x^3}} = \lim_{x\to+\infty} \frac{2x\cdot(-2)}{x^4-1}\cdot\left(-\frac{x^3}{2}\right) = \lim_{x\to+\infty} \frac{2x^4}{x^4-1} = 2$

$\Longrightarrow \lim_{x\to+\infty} \left(\frac{x^2+1}{x^2-1}\right)^{x^2} = e^2$; $\square$

14. (g).

$\lim_{x\to 0} \left(\frac{\sin(x)}{x} \right)^{\frac{1}{x^2}}$

consider again $\lim_{x\to 0} \frac{\ln\left(\frac{\sin x}{x}\right)}{x^2} \overset{\left(\frac{0}{0}\right)}{\overset{H}{=}} \lim_{x\to 0} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x}$

$= \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^2 \sin x} \overset{\text{substitution}}{=\!=} \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^3} \cdot \frac{x}{\sin x}$ $\to 1$

$\overset{H}{=} \lim_{x\to 0} \frac{\cancel{\cos x} + x\cdot(-\sin x) - \cancel{\cos x}}{6x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = -\frac{1}{6}$;

$\Downarrow$

($\overset{H}{=}$ is $=$)

Hence $\lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}$; $\square$

Tutorial 8

Topics : L'Hôpital Rule & Taylor's Theorem

Q1) [L'Hôpital] Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable.

If $f(0) = g(0) = 0$ and $\displaystyle\lim_{x \to 0} \frac{f'(x)}{g'(x)} = L$ exists

Show that $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = L$

Q2) Evaluate the limits

a) $\displaystyle\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x}$ b) $\displaystyle\lim_{x \to 0^+} x^{\frac{1}{1 + \ln x}}$

c) $\displaystyle\lim_{x \to 0} \frac{1}{x^{k+1}} \left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!} \right)$

Q3) find the taylor's Polynomial up to degree 2 at centre $x=c$.

a) $f(x) = (1+\sin x)^2$; $c = 0$

b) $f(x) = \ln x$; $c = e$

Q4) Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable

if $f(x) = 1 + a_1 x + a_2 x^2 +$ higher order term

find the taylor polynomial of $\frac{1}{f}$ up to degree 2.

Recall:

L'Hôpital Rule: Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable

If $\lim_{x \to c} f(x), \lim_{x \to c} g(x) = 0$ or $\pm\infty$ for $c \in \mathbb{R}$ or $c = \pm\infty$

and $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ exists Then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$.

Taylor's Theorem Let $k = 1, 2, \dots$

If $f : \mathbb{R} \to \mathbb{R}$ is k-times differentiable at $x = x_0$

then $f(x) = P_k(x) + R_k(x) \quad \forall x \in (x_0 - \delta, x_0 + \delta) \quad \exists \delta > 0$

s.t. $\begin{cases} P_k(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \dots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ \lim_{x \to x_0} \frac{R_k(x)}{(x - x_0)^k} = 0 \end{cases}$

$\underline{\text{Sol}^n}$

Q1) since $\lim_{x\to 0} \frac{f'}{g'}$ exists $\Rightarrow g \not\equiv 0$, let $x \in \mathbb{R}\setminus\{0\}$

By Cauchy's MVT. $\exists\, c_x \in (\min\{0,x\}, \max\{0,x\})$ s.t.

$$\frac{f'(c_x)}{g'(c_x)} = \frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f(x)}{g(x)}$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(c_x)}{g'(c_x)} = \lim_{\tilde{x}\to 0} \frac{f'(\tilde{x})}{g'(\tilde{x})}$$

since $\lim_{x\to 0} c_x = 0$

Q2) a) $$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{2\sin x \cos x}{-\sin x}$$

$$= \lim_{x\to 0} (-2\cos x) = -2 \,//$$

b) $$\lim_{x\to 0} x^{\frac{1}{1+\ln x}} = \lim_{x\to 0} e^{\ln\left(x^{\frac{1}{1+\ln x}}\right)}$$

$$= \lim_{x\to 0} e^{\frac{\ln x}{1+\ln x}} = e^{\lim\limits_{x\to 0} \frac{\ln x}{1+\ln x}} \qquad \left(\frac{-\infty}{-\infty}\right)$$

$$= e^{\lim\limits_{x\to 0} \frac{1/x}{1/x}} = e^1 = e \,//$$

2c)

$$\lim_{x\to 0} \frac{e^x - 1 - x - \cdots - \frac{x^k}{k!}}{x^{k+1}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^x - 1 - \cdots - \frac{x^{k-1}}{(k-1)!}}{(k+1)x^k} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^x - 1 - \cdots - \frac{x^{k-2}}{(k-2)!}}{(k+1)(k)x^{k-1}} \quad \left(\frac{0}{0}\right)$$

$$\vdots$$

$$= \lim_{x\to 0} \frac{e^x}{(k+1)!} = \frac{1}{(k+1)!}$$

//

3a) given $f(x) = (1+\sin x)^2$

by Taylor thm $f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 +$ higher order terms.

$f(x) = (1+\sin x)^2 \Rightarrow f(0) = 1$

$f'(x) = 2(1+\sin x)\cos x \Rightarrow f'(0) = 2$

$f''(x) = 2(1+\sin x)(-\sin x) + 2\cos^2 x \Rightarrow f''(0) = 2$

Hence

$f(x) = 1 + 2x + x^2 +$ higher order terms.

3b) Given $f(x) = \ln x$, centre $= e$.

by taylor thm: $f(x) = f(e) + f'(e)(x-e) + \frac{f''(e)}{2}(x-e)^2 + \cdots$

$f(x) = \ln x \Rightarrow f(e) = 1$

$f'(x) = \frac{1}{x} \Rightarrow f'(e) = \frac{1}{e}$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(e) = \frac{-1}{e^2}$

Hence $f(x) = 1 + \frac{1}{e}(x-e) + \frac{-1}{2e^2}(x-e)^2 + \cdots$

4) Since $f(x) = 1 + a_1 x + a_2 x^2 + \cdots$ is twice differentiable

and $\frac{1}{f}$ is also twice differentiable (Verify by quotient rule)

Suppose $\frac{1}{f}(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$$1 \equiv f \cdot \frac{1}{f} = (1 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= b_0 + (b_1 + a_1 b_0)x + (b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

$$\Rightarrow \begin{cases} 1 = b_0 \\ 0 = b_1 + a_1 b_0 \\ 0 = b_2 + a_1 b_1 + a_2 b_0 \end{cases} \Rightarrow \begin{cases} b_0 = 1 \\ b_1 = -a_1 b_0 = -a_1 \\ b_2 = -a_1 b_1 - a_2 b_0 = a_1^2 - a_2 \end{cases}$$

Hence $\frac{1}{f}(x) = 1 + (-a_1)x + (a_1^2 - a_2)x^2 + \cdots$

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Note Title 10/27/2014

* More examples for L'Hopital's Rule

$$\lim_{x\to(\frac{\pi}{4})^-} (\tan x)^{\tan(2x)}$$

$$f(x)^{g(x)} = e^{g(x)\ln f(x)}$$

write $(\tan x)^{\tan(2x)} = \boxed{e^{\tan(2x)\ln(\tan x)}}$

$$= \lim_{x\to(\frac{\pi}{4})^-} e^{\tan(2x)\ln(\tan x)}$$

Since e^x is cont's on $\mathbb{R}$

$$= e^{\lim_{x\to\frac{\pi}{4}^-} \tan(2x)\ln(\tan x)}$$

$$\lim_{x\to\frac{\pi}{4}^-} \tan(2x)\ln(\tan x)$$

$2x \to \frac{\pi}{2}$, $\tan x \to 1$, $\ln(\tan x) \to 0$

$\underline{\infty \cdot 0}$

$$= \lim_{x\to\frac{\pi}{4}^-} \frac{\ln(\tan x)}{\frac{1}{\tan 2x}}$$

$\tan x = \frac{\sin x}{\cos x}$

$$= \lim_{x\to\frac{\pi}{4}^-} \frac{\frac{1}{\tan x}\cdot\frac{1}{\cos^2 x}}{\frac{1}{(\tan 2x)^2}\cdot\left(-\frac{2}{\cos^2 2x}\right)}$$

$$= \lim_{x\to\frac{\pi}{4}^-} \frac{\frac{1}{\sin x\cos x}}{-\frac{2}{\sin^2 2x}} = \frac{\frac{2}{\sin 2x}}{-\frac{2}{\sin^2 2x}} = -1$$

use $\sin 2x = 2\sin x\cos x$

$$\boxed{\cos x \to \frac{\sqrt{2}}{2},\ \sin x \to \frac{\sqrt{2}}{2} \quad \sin 2x \to 1}$$

So $\lim_{x\to\frac{\pi}{4}^-} (\tan x)^{\tan(2x)} = e^{-1}$

* Taylor's thm :

$f:(a,b)\to\mathbb{R}$ has $(n+1)$th derivatives, then $\forall c\in(a,b)$

$$f(x) = \underbrace{f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n}_{P_n(x) = \text{degree } n \text{ Taylor polynomial}} + \underbrace{\frac{f^{(n+1)}(\xi)(x-c)^{n+1}}{(n+1)!}}_{E_n(x) = \text{error term}}$$

for some ξ between x and c

(note this ξ depends on x and c.)

- Taylor polynomial is used to approximate function $f(x)$

Example: Find Taylor polynomial of $f(x)=\ln x$, centered at $c=e$, up to deg 3.

Sol'n: $f(c) = \ln e = 1$,

$f'(x) = \frac{1}{x}$ $\quad f'(c) = \frac{1}{e}$

$f''(x) = -\frac{1}{x^2}$, $\quad f''(c) = -\frac{1}{e^2}$

$f'''(x) = \frac{2}{x^3}$, $\quad f'''(c) = \frac{2}{e^3}$

So $P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3$$

$\ln(3) = 1.09861228867$, $\quad \ln(4) = 1.38629436112$

$P_3(3) = 1.09863892883$, $\quad P_3(4) = 1.39529728823$

$P_2(3) = 1.09826787246$ $\quad P_2(4) = 1.36035326348$

Approximation:

Generally, accuracy $\uparrow$ as deg n $\uparrow$

as $|x-c|$ $\downarrow$

This is explained by the error term

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)\,(x-c)^{n+1}}{(n+1)!}$$

when $|x-c|<1$, $n \nearrow$, then $E_n(x) \searrow$

$|x-c| \searrow$, then $E_n(x) \searrow$.

Example: Approximate $f(x) = \cos x$, at $x=1$, accurate to 3 decimal places.

$\boxed{\text{center ? degree ?}}$

$\rightsquigarrow$ means: maximal possible error $< 10^{-3}$

$\rightsquigarrow$ i.e. choose center c, find n, s.t. $|E_n(1)| < 10^{-3}$

① choose c s.t. $\begin{cases} |1-c| \text{ is small} \\ f^{(n)}(c) \text{ is easy to compute} \end{cases}$

in this case, may take

$c = \frac{\pi}{3}$, since $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, $\cos(\frac{\pi}{3}) = \frac{1}{2}$

and $|1 - \frac{\pi}{3}| < 1$

② Estimate a maximal error, i.e. estimate $|E_n(1)| < L(n)$

then let $L(n) < 10^{-3} \Rightarrow n > ?$

$$|E_n(x)| = \left|\frac{f^{(n+1)}(\xi)\,(x-c)^{n+1}}{(n+1)!}\right| = \left|\frac{(x-c)^{n+1}}{(n+1)!}\right| \left|f^{(n+1)}(\xi)\right|$$

In our case, $f^{(4k)}(x) = \cos x$, $f^{(4k+1)}(x) = -\sin x$

$f^{(4k+2)}(x) = -\cos x$, $f^{(4k+3)}(x) = \sin x$

So $\left|f^{(n+1)}(\xi)\right| \leq 1$

$$|E_n(1)| \leq \left|\frac{(1-\frac{\pi}{3})^{n+1}}{(n+1)!}\right|$$ use calculator

let $\left|\frac{(1-\frac{\pi}{3})^{n+1}}{(n+1)!}\right| < 10^{-3} \Rightarrow n > 2$ at least 3.

$\Rightarrow$ deg 3 Taylor polynomial centered at $\frac{\pi}{3}$

$$P_3(x) = f(\tfrac{\pi}{3}) + f'(\tfrac{\pi}{3})(x-\tfrac{\pi}{3}) + \frac{f''(\frac{\pi}{3})}{2!}(x-\tfrac{\pi}{3})^2 + \frac{f'''(\frac{\pi}{3})}{3!}(x-\tfrac{\pi}{3})^3$$

$$= \cos\tfrac{\pi}{3} - \sin\tfrac{\pi}{3}(x-\tfrac{\pi}{3}) - \frac{\cos\frac{\pi}{3}}{2}(x-\tfrac{\pi}{3})^2 + \frac{\sin\frac{\pi}{3}}{6}(x-\tfrac{\pi}{3})^3$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}(x-\tfrac{\pi}{3}) - \frac{1}{4}(x-\tfrac{\pi}{3})^2 + \frac{\sqrt{3}}{12}(x-\tfrac{\pi}{3})^3$$

So $P_3(1) = \boxed{0.5403022}008$ ← approximate value

$\cos(1) = \boxed{0.5403023}0586$

$E_3(1) = \underline{1.0506 \times 10^{-7}} < 10^{-3}$

Summary: two type of Qs related to approximation.

① write down $\underline{\text{degree } n}$ Taylor polynomial $\underline{\text{centered at } c}$

② Approximate $f(x)$ at $x=a$, w/ accuracy up to $\underline{k\text{-decimal pts}}$

⇝ Take a c to be your center $\begin{cases} |a-c| \text{ small} \\ f^{(n)}(c) \text{ can be calculated} \end{cases}$

⇝ Estimate $|E_n(a)| < L(n)$ ← not depend on ξ

let $\underline{L(n) < 10^{-k}}$

$\Rightarrow$ find a minimal n

write down $P_n(x)$ centered at c.

$P_n(a)$ ← approximate value.

Exercise :

(1) $\lim_{x \to \frac{\pi}{2}^-} (\tan x - \sec x)$

(2) $\lim_{x \to 0^+} x^x$

(3) Approximate $e^{1.5}$ w/ accuracy of 3-decimal pts.